Eigen Vectors and Eigen values

CONTINUED

Let V and W be vector spaces over the same field F.

Linear Transformation

These are the analogue for vector spaces of a homomorphism of groups, namely

$$T : V \longrightarrow W$$

which is compatible with addition and scalar multiplication i.e.

$$T(v_1+v_2) = T(v_1)+T(v_2)$$

$$\& \quad T(kv) = kT(v) \quad , v_1, v_2, v \in V, \; k \in F$$

is called a linear transformation.

Note that $T(\sum_i \alpha_i v_i) = \sum_i \alpha_i T(v_i)$

Example. (Left Multiplication by a Matrix)

Let A be an $m \times n$ matrix with entries in F. Consider A as an operator on column vectors:

$$A : F^n \longrightarrow F^m$$
$$X \rightsquigarrow AX, \quad \forall X \in F^n$$

Note that $A_{m \times n} \cdot X_{n \times 1} \in F^m$

Linear Operators: A linear transformation $T: V \to V$ of a vector space to itself is called a linear operator on V.

Note that if A is an $n \times n$ matrix, then A defines a linear operator on the space F^n of column vectors. (in the above example).

2.3

Recall, eigenvalues of a matrix are the zeros of its characteristic polynomial.

Example. Consider the 2×2 matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.
Then the characteristic polynomial of A is –

$$\det(A - \lambda I_2) = \left| \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$$

$$= \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4$$

$$= \lambda^2 - 2\lambda - 3$$

$$= (\lambda - 3)(\lambda + 1)$$

Thus in view of Theorem 1.6, eigenvalues of A are 3 and −1.

1.9. Remark: Similar matrices have the same characteristic polynomials.

Suppose A and B are similar matrices. Then there exists an invertible matrix P such that

$$B = P^{-1} A P. \qquad \text{--- (1)}$$

We show that A and B have same characteristic polynomial.

Now, the characteristic polynomial of B

$$|\lambda I - B| = |\lambda I - P^{-1} A P| \quad \text{--- from (1)}$$

$$= |P^{-1}\lambda I P - P^{-1} A P| \quad (\because \lambda I = P^{-1}\lambda I P)$$

$$= |P^{-1}(\lambda I - A) P|$$

$$= |P^{-1}||\lambda I - A||P|$$

$$= |\lambda I - A| = \text{characteristic polynomial of } A.$$

using the fact that determinants are scalars and commutative, Also $|P^{-1}||P| = 1$.

Hence the result.

Example Let us consider the 2×2 matrix

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}.$$

and the standard basis vector $e_1 = (1,0) \in \mathbb{R}^2$. Then

$$Ae_1 = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$= 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3e_1$$

shows that e_1 is an eigenvector for the linear operator, left multiplication by matrix, note that the eigenvalue associated with the eigenvector e_1 is 3.

Example. Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

Let $v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (0,1,1)^t \in \mathbb{R}^3$. Find the eigenvalue by proving that v is an eigenvector of the linear operator, left multiplication by the matrix A; i.e.

$$A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$v_0 \rightsquigarrow A.v_0. \quad \forall v_0 \in \mathbb{R}^3.$$

DO it yourself.

Characteristic Polynomial of a Linear Operator –

Let V be a vector space over the field F. Let $\mathcal{B}$ and $\mathcal{B}'$ be two ordered basis of V, such that

$$[T]_{\mathcal{B}} = A \quad \text{and} \quad [T]_{\mathcal{B}'} = B.$$

Then $B = P^{-1}AP$ for some invertible matrix P.

Let $T: V \longrightarrow V$ be a linear operator. A scalar $\lambda \in F$ is an eigen value of T

iff $T - \lambda I$ is singular

iff $\det [T - \lambda I]_{\mathcal{B}} = 0$

iff $\det\left([T]_{\mathcal{B}} - \lambda I\right) = 0$.

iff $\det (A - \lambda I) = 0$

iff λ is an eigen value of A.

As both the matrices A and B are similar, so λ is an eigen value of A iff λ is an eigen value of B.

Thus, $\lambda \in F$ is an eigen value of T iff λ is an eigen value of corresponding matrix of T w.r.t. any basis of V.

If $[T]_{\mathcal{B}} = A$, then the characteristic polynomial of T is $|\lambda I - A|$.

Remark: Let $T : R^2 \longrightarrow R^2$ be a linear operator which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then T has no eigen values in R.

For the characteristic polynomial of T (or A) i.e.

$$\Delta(\lambda) = |\lambda I - A|$$

$$= \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

which has no real roots. So A does not have any eigen values in R.

However, $\lambda^2+1=0 \Rightarrow \lambda = \pm i \in C$, so $\pm i$ are eigen values of A in the field of complex number C.

Remark: Eigen values of an $n \times n$ triangular matrix A are the diagonal entries of A.