FRAUNHOFER DIFFRACTION

B.Sc. II (Paper I)

DR. A.K. DWIVEDI

DEPARTMENT OF PHYSICS

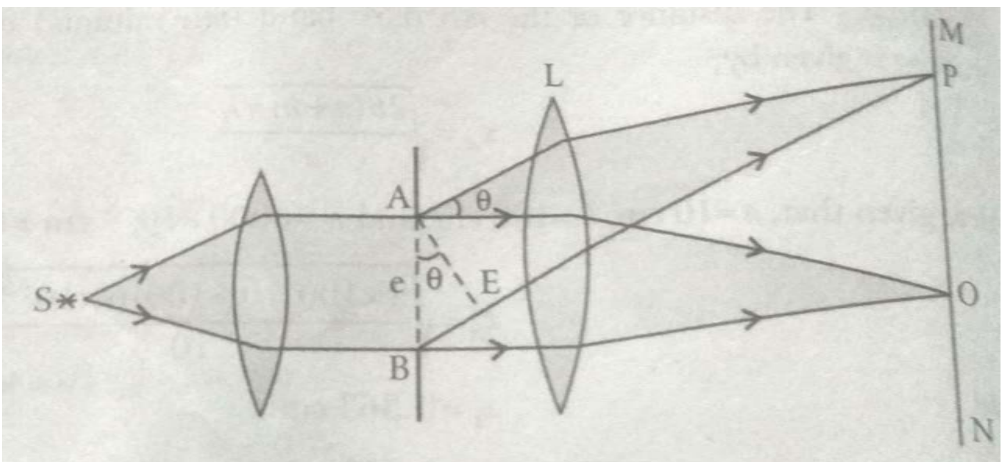
HARISH CHANDRA P.G. COLLEGE VARANASI

FRAUNHOFER DIFFRACTION AT A SINGLE SLIT

In FRAUNHOFER DIFFRACTION the source and the screen are a infinite distance from the obstacle and the wavefront is plane.

Let a parallel beam of monochromatic light of wavelength be incident normally upon a narrow slit AB of width e where it gets diffraction in below fig. if a lens L is placed in the path of the diffraction beam, a real image of the diffraction pattern is formed on the screen MN in focal plane of the lens.

$$BE = AB \sin\theta = e \sin\theta$$



The corresponding phase difference= $(\frac{2\pi}{\lambda})$*path difference

$$= (\frac{2\pi}{\lambda})*(e \sin\theta)$$

Now, consider the width AB of the slit divided into n equal parts. Each part forms an elementary source. The amplitude of vibration at P due to the wave from each part will be the same, and the phase difference the waves from any two consecutive parts is,

$$= \frac{1}{n}(\frac{2\pi}{\lambda} e \sin\boldsymbol{\theta}) = d \text{ (let)}$$

Hence, resultant amplitude at P is given by,

$$A = \frac{a \sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)}$$

$$=\frac{a\sin\left(\frac{\pi e\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi\, e\sin\theta}{n\lambda}\right)}$$

Let $\frac{\pi e\sin\theta}{\lambda}=\boldsymbol{\alpha, then}$

$$A=\frac{a\sin\alpha}{\sin\left(\frac{\alpha}{n}\right)}=\frac{a\sin\alpha}{\alpha/n}=\frac{na\,\sin\alpha}{\alpha}$$

$$\left(\because \frac{\alpha}{n}\ \textit{is very small}\right)$$

If n→ ∞, $\boldsymbol{\alpha\to 0}$, ***but*** the product na remains finite.let na=A_0, then

$$\mathbf{A}=\frac{A0\ \sin\alpha}{\alpha}$$

So the resultant intensity at P will be,

$$\mathbf{I=A^2\ or\ I=}\left(\frac{A0\ \sin\alpha}{\alpha}\right)^2 \qquad \ldots..\ (1)$$

Since, the magnitude intensity at any point in the focal plane of the lens is a function of $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}$.so, we obtain a series of maxima and minima

Condition For Minima

From eq. (1) it is clear that the intensity is zero, when

$$\text{Sin}\boldsymbol{\alpha}=0$$

$$\left[\text{but } \boldsymbol{\alpha\neq 0},\frac{\sin\alpha}{\alpha}=\mathbf{1},\boldsymbol{When\ \alpha=0}\right]$$

or $\boldsymbol{\alpha}=\pm n\boldsymbol{\pi}$

Where n is the integer except zero.

But, we know that,

$$\boldsymbol{\alpha}=\frac{\pi e\sin\theta}{\lambda}$$

So, the position of minima are, given by

$$\boldsymbol{\pi e\sin\theta/\lambda}=\pm\boldsymbol{n\pi}$$

Or $\boldsymbol{e\sin\theta=\pm n},=\pm\boldsymbol{\lambda}\ ,\pm\mathbf{2}\boldsymbol{\lambda}\ ,\pm\ \mathbf{3}\boldsymbol{\lambda}\ ,\boldsymbol{and\ so\ on}$

The eq. given the directions of first, second, third,…. and so on minima.

Condition For Maxima

To locate the position of maxima of intensity in the diffracting pattern, let us differentiate I with respect to α and equal to zero i.e.,

$$\frac{dI}{d\alpha}=0$$

$$\frac{d\left(\frac{A0\ \sin\alpha}{\alpha}\right)^2}{d\alpha}=0$$

$$A_0^2\ \left[\frac{\alpha^2\ \ 2\sin\alpha\cos\alpha-\sin^2\alpha\ .2\alpha}{\alpha^4}\right]=0$$

$$\boldsymbol{\alpha^2\ \ \sin\alpha\cos\alpha-\sin^2\alpha\ .\ \alpha}=0$$

$$\alpha \sin \alpha[\alpha \cos \alpha - \sin \alpha] = 0$$

$$\alpha \sin \alpha = 0$$
$$\alpha \cos \alpha = [\sin \alpha]$$

$$\alpha = \tan \alpha$$

This eq. is solved graphically by plotting the

$$Y= \alpha \qquad \text{..... (2)}$$
$$Y=\tan \alpha \qquad \text{...... (3)}$$

The point of intersection of these two curves given the roots of eq. $\alpha = \tan \alpha$**. Eq.(2)** and (3) are shown in graph.

$$\alpha = 0, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi\ldots}{2}.....$$

Or in general,

Substituting these value of in the equation(1), we get

$$I_{0_{\alpha\to 0}} = \left(\frac{A0 \sin 0}{0}\right)^2 = A_0{}^2$$

$$I_{0_{\alpha\to\frac{3\pi}{2}}} = \left(\frac{A0 \sin\frac{3\pi}{2}}{\frac{3\pi}{2}}\right)^2 = \frac{4}{9\pi^2}A_0{}^2 = \frac{A_0{}^2}{22}$$

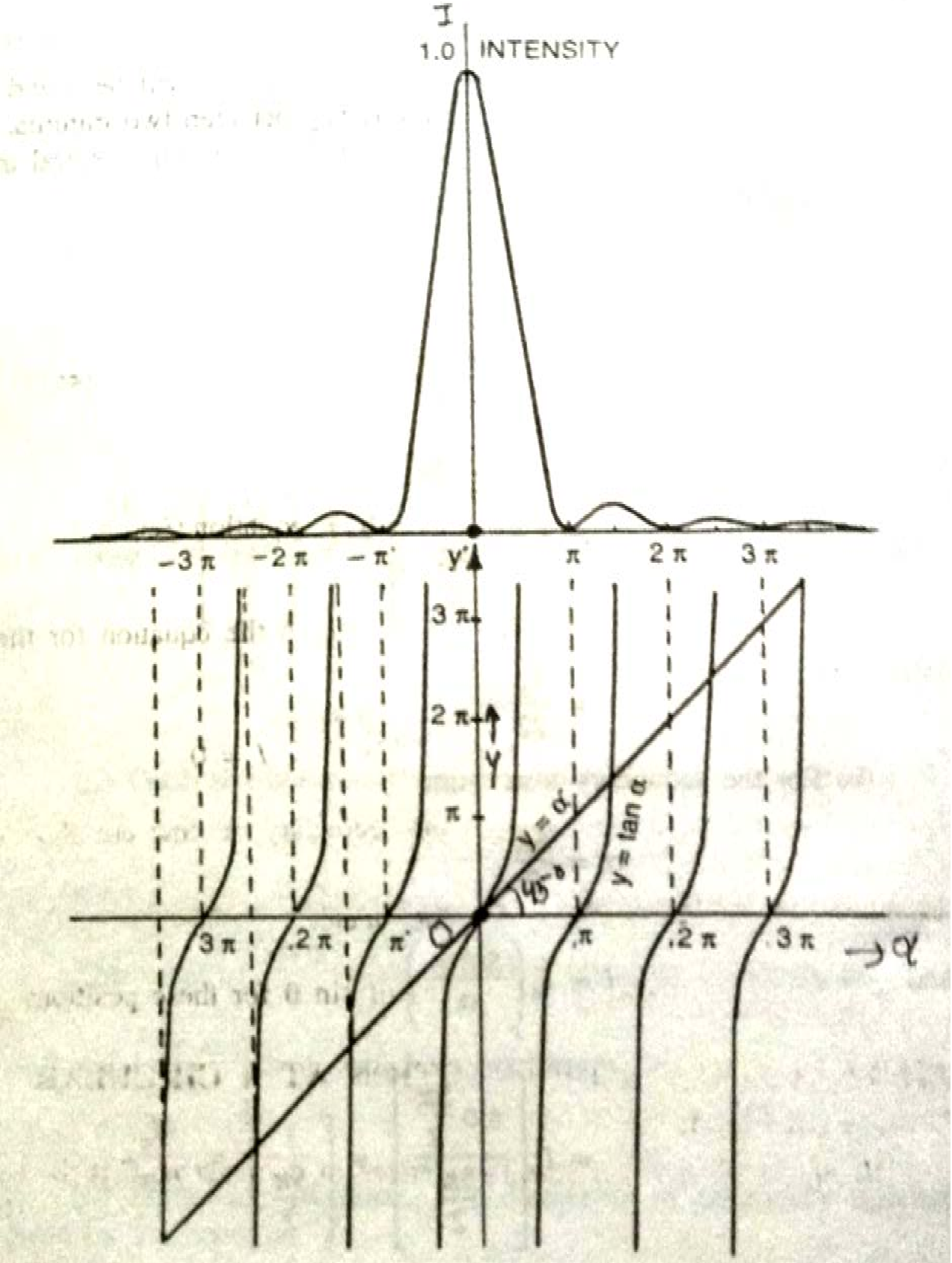
$$I_{0_{\alpha\to\frac{5\pi}{2}}} = \left(\frac{A0 \sin\frac{5\pi}{2}}{\frac{5\pi}{2}}\right)^2 = \frac{4}{25\pi^2}A_0{}^2 = \frac{A_0{}^2}{61}$$

Thus, the intensity of the successive maxima are in the ratio

$$1:\frac{1}{22} : \frac{1}{61} : \frac{1}{121} \ldots\ldots.$$

Cleary, most of the incident light is concentrated in the principal maxima which occurs in the direction given by $\alpha = 0$

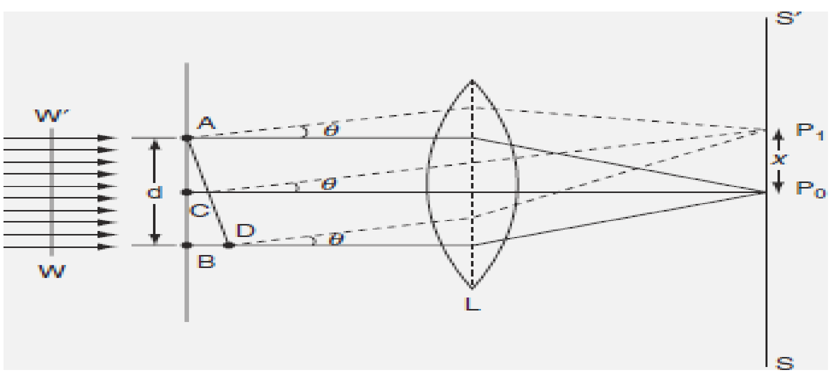
$$\frac{\pi e \sin\alpha}{\lambda} = 0 \qquad \textbf{and} \qquad \theta = 0$$


I
1.0
INTENSITY
−3π
−2π
−π
y'
π
2π
3π
3π
2π
π
y
y = α
y = tan α
O
3π
2π
π
π
2π
3π
α

i.e.,in the direction of the incident light. Thus, the diffraction pattern consist of direction of the incident light, having maxima on either side of it, as shown in above fig. the maxima lie at $\boldsymbol{\alpha = \pm\pi, \pm 2\pi, \pm 3\pi}$ … … …
The weak maxima do not fall exactly mid way between two minima, but are displaced towards the centre of the pattern by a certain amount which decreases with increasing order.

Fraunhofer Diffraction At A Circular Aperture

The problem of diffraction at a circular aperture was first solved by Airy. A circular aperture of diameter'*d*' is shown as *AB* in below Fig. A plane wave front *WW'* is incident normally on this aperture. Every point on the plane wave front in the aperture acts as a source of secondary wavelets. The secondary wavelets spread out in all directions as diffracted rays in the aperture. These diffracted secondary wavelets are converged on the screen *SS'* by keeping a convex lends (*L*) between the aperture and the screen. The screen is at the focal plane of the convex lens. Those diffracted rays traveling normal to the plane of aperture [i.e., along CP_0] are get converged at P_o.



All these waves travel some distance to reach P_o and there is no path difference between these rays. Hence a bright spot is formed at P_o known Airy's disc. P_o corresponds to the central maximum.

Next consider the secondary waves traveling at an angle θ with respect to the direction of CP_o. All these secondary waves travel in the form of a cone and hence, they form a diffracted ring on the screen. The radius of that ring is x and its centre is at P_o. Now consider a point P_1 on the ring, the intensity of light at P_1 depends on the path difference between the waves at *A* and *B* to reach P_1. The path difference is $BD = AB \sin\theta = d \sin\theta$. The diffraction due to a circular aperture is similar to the diffraction due to a single slit. Hence, the intensity at P_1 depends on the path difference $d \sin \theta$. If the path difference is an integral multiple of λ then intensity at P_1 is minimum. On the other hand, if the path difference is in odd multiples of $\lambda/2$, then the intensity is maximum.

i.e. , $d \sin \theta = n \lambda$ for minima …… (1)

and $d \sin \theta = (2n-1)\lambda/2$ for maxima ….. (2)

where n = 1, 2, 3, …. etc. n = 0 corresponds to central maximum.

The Airy disc is surrounded by alternate bright and dark concentric rings, called the Airy's rings. The intensity of the dark ring is zero a and the intensity of the bright ring decreases as we go radially from P_0 on the screen. If the collecting lens (L) is very near to the circular aperture or the screen is at a large distance from the lens, then

$$\sin \theta \approx \theta \approx x/f \quad \text{.....} \quad (3)$$

Where f is the focal length of the lens.

Also from the condition for first secondary minimum [using equation (1)]

$$\sin \theta \approx \theta \approx \lambda/d \quad \text{....} \quad (4)$$

Equations (3) and (4) are equal

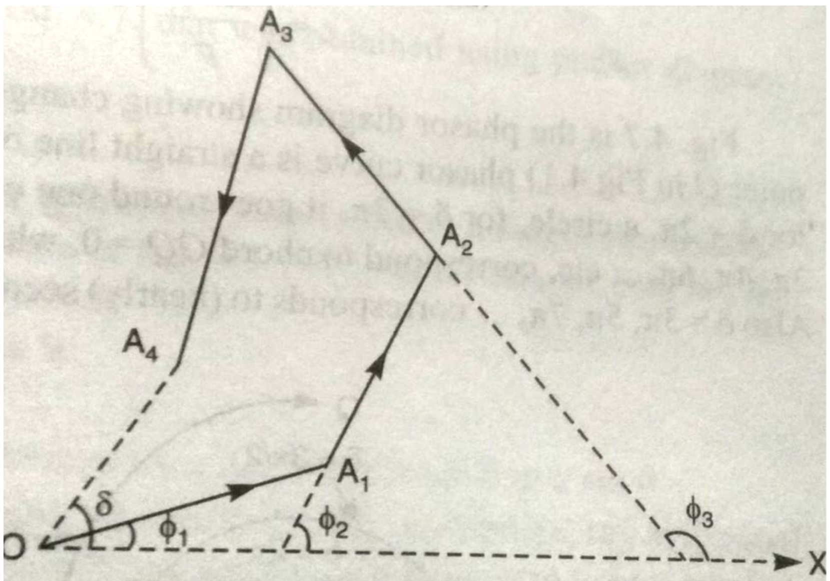
So, $x/f = \lambda/d$ or $x = f\lambda/d$

But according to Airy, the exact value of x

$$\mathbf{x = 1.22 f \lambda/d} \quad \text{....} \quad (5)$$

using equation (5) the radius of Airy's disc can be obtained. Also from equation (5) we know that the radius of Airy's disc is inversely proportional to the diameter of the aperture. Hence by decreasing the diameter of aperture, the size of Airy's disc increases.

The Phosar Diagram Method For Fraunhhofer Diffraction At A Slit



In fig. given the case of addition several simple harmonic vibrations of same frequency. We can select any reference phase line OX, and successive draw length OA_1, A_1A_2, A_2A_3, A_3A_4, ….. proportional to the amplitude of the constitutions vibrations and in direction making angles with OX equal to their phase angles $\emptyset_1$, $\emptyset_2$, $\emptyset_3$, ……etc . The resultant vibration is shown in amplitude and phase by closing the last arm of the polygon taken in opposite sense. All positive $\emptyset's$ must be measured anticlock wise from $\emptyset's$.

The addition of the secondary wavelets from different part of slit AB in below fig. (1) as they reach P can b e obtained by the phasor diagram method.

Let us divided the slit AB into p equal parts. Then the amplitude contributed by each element is A_0/p, if A_o is that if the entire slit sends waves in the same phase. Also, the phase difference between waves for successive waves elements is δ/p, where

$$\boldsymbol{\delta = \frac{2\pi}{\lambda}(a \sin \theta)}$$

Fig. shown the phosor diagram to add p disturbances of amplitude A0/p each and successive phase disturbance $\frac{\delta}{p}$. fig. is the result as p term to infinity; the phasor diagram become a smooth circular arc OQ with δ as $s$hown.

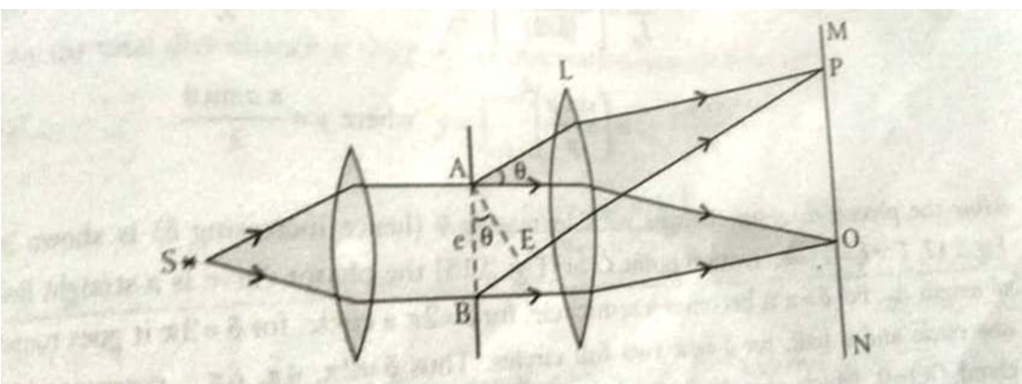


Fig. (1)

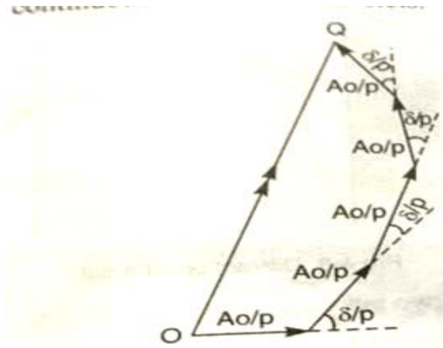


Fig.(2)

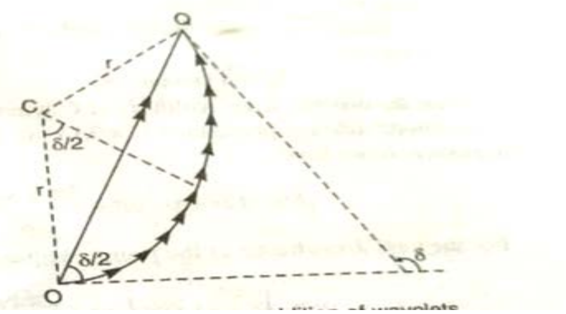


fig.(3)

The resultant amplitude AO at the observation point P given by the chord OQ in fig. (2). We thus have

$$\frac{A_\theta}{A_0} = \frac{chord\ OQ}{Arc\ OQ} = \frac{2r\sin(\frac{\delta}{2})}{2r.\frac{\delta}{2}} = \frac{\sin\frac{\delta}{2}}{\left(\frac{\delta}{2}\right)} \qquad \textbf{..... (2),}$$

For intensities, we then get

$$\frac{I_\theta}{I_0} = \left[\frac{\sin\frac{\delta}{2}}{\left(\frac{\delta}{2}\right)}\right]^2 \qquad \text{...... (3)}$$

$$= \left(\frac{\sin p}{p}\right)^2 \qquad \textbf{.....(4)}$$

where $\delta = \frac{2\pi \sin\theta}{\lambda}$ and Where p=$\frac{\pi \sin\theta}{\lambda}$

In below Fig.(4) is the phasor diagram showing change in δ with increasing θ. for δ =0 phasor curve is a straight line of length A_0. For$\delta = \pi$, the phosor curve is a straight line of length A_0. For $\delta = \pi$ it become a semicircle, for $\delta = 2\pi$ it goes round one circle and a half, for two full circle .thus, $\delta = 2\pi, 4\pi, 6\pi$ .... correspond to chord OQ=0, which means zero amplitude. Also , $\delta = 3\pi, 5\pi, 7\pi$ …correspond to secondary maxima, the main maxima being at $\delta = 0$

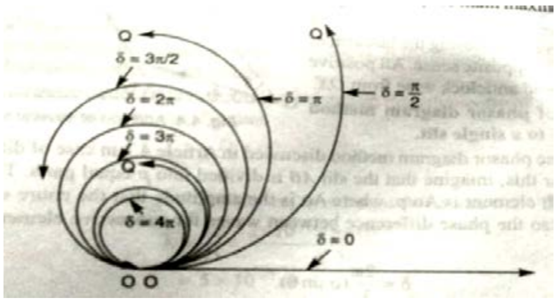

Q
$\delta = 3\pi/2$
$\delta = 2\pi$
$\delta = 3\pi$
Q
$\delta = 4\pi$
Q
$\delta = \pi$
$\delta = \frac{\pi}{2}$
$\delta = 0$
O O

Fig.(4)

.

References:

1. A textbook of Optics by Brij Lal and Subramaniam
2. Optics by Ajay Ghatak
3. Physical Optics and Lasers by J.P. Agrawal
4. Physical Optics and Lasers by Tripathi and Singh