# B.Sc 3$^{rd}$ YEAR

## PAPER - 1$^{ST}$ , UNIT- (IV)
## { METRIC SPACE }

**METRIC:**

Let $X \neq \phi$ be set then a function $d : X \times X \rightarrow R$ ( set of real numbers) is called metric on X if it satisfies the following conditions -

(i) $d(x, y) \geq 0$ (Non-negativity)

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$ (Symmetry)

(iv) $d(x, y) \leq d(x, z) + d(z, y), \forall\ x, y, z \in X$ (Triangle inequality)

**Note**: d is also called distance function.

**METRIC SPACE:**

The pair (X, d) is called metric space.

**EXAMPLE:** (1)

Let X=R, $d : X \times X \rightarrow R$ defined by $d(x, y) = |x - y|$ then d is a metric on X.

**SOLUTION:**

(i) let $x, y \in R$ then $x-y \in R \implies |x - y| \geq 0 \implies d(x, y) \geq 0$

(ii) $d(x, y)= 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x = y$

(iii) $d(x, y)= |x - y| = |-(y-x)| = |y - x| = d(y, x)$

(iv)let $z \in R$, $d(x, y)= |x - y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x, z) + d(z, y)$
{by triangle inequality in modulus}

Thus d is a metric on R .

REMARK: The above metric is called **usual metric on R**.

**EXAMPLE:** (2) Let X be a non empty s et and $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases} \quad \forall\ x, y \in X$

then d is a metric and is called **discrete meric on X.**

**SOLUTION:**

(i) From the definition of d , for all $x , y \in X$ , d(x, y) is either 0 or 1 hence $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$ ( by def. of d )

(iii) let $x, y \in X$ then $x = y$ or $x \neq y$

if $x = y \implies y = x$, so $d(x, y) = d(y, x)$

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if $x \neq y \Longrightarrow y \neq x$, so $d(x, y) = d(y, x)$
in both cases $d(x, y) = d(y, x)$

(iv) let $z \in X$
case I: if $x = y$ then either $x = y = z$ or $x = y \neq z$
$\Longrightarrow$ either $d(x, y) = d(x, z) = d(z, y) = 0$ or $d(x, y) = 0$ and $d(x, z) = d(z, y) = 1$
$\Longrightarrow$ either $d(x, y) = d(x, z) + d(z, y)$ or $d(x, y) < d(x, y) + d(z, y)$
so $d(x, y) \leq d(x, y) + d(z, y)$
cases II: if $x \neq y$ then either $x \neq y = z$ or $x \neq y \neq z$
$\Longrightarrow$ either $d(x, y) = d(x, z) = 1$ and $d(z, y) = 0$ or $d(x, y) = d(x, y) = d(z, y) = 1$
$\Longrightarrow$ either $d(x, y) = d(x, y) + d(z, y)$ or $d(x, y) < d(x, y) + d(z, y)$
so, $d(x, y) \leq d(x, y) + d(z, y)$
thus in either case $d(x, y) \leq d(x, y) + d(z, y)\ \forall\ x, y, z \in X$, hence d is a metric on X

**NOTE:**

**MINKOSWKI INQUALITY:** if $p \geq 1$, $x_i, y_i$ are positive real numbers $\forall\ i \in N$

$$\left[\sum_{i=1}^{n}(x_i + y_i)^p\right]^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} y_i^p\right)^{1/p}$$

**EXAMPLE:** (3) Let $X = R^2$, $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$ then d is a metric on $R^2$.

**SOLUTION:**

(i) since $(x_1 - y_1)^2 \geq 0$, $(x_2 - y_2)^2 \geq 0 \Longrightarrow (x_1 - y_1)^2 + (x_2 - y_2)^2 \geq 0$
$\Longrightarrow [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} \geq 0 \Longrightarrow d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} = 0 \Leftrightarrow (x_1 - y_1)^2 + (x_2 - y_2)^2 = 0$
$\Leftrightarrow (x_1 - y_1)^2 = 0$ and $(x_2 - y_2)^2 = 0 \Leftrightarrow x_1 - y_1 = 0$ and $x_2 - y_2 = 0 \Leftrightarrow x_1 = y_1$ and $x_2 = y_2$
$\Leftrightarrow x = y$

(iii) $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} = [(y_1 - x_1)^2 + (y_2 - x_2)^2]^{1/2} = d(y, x)$

(iv) $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2} = \left[\sum_{i=1}^{2}(x_i + y_i)^2\right]^{1/2}$

let $z = (z_1, z_2)$ then $d(x, y) = \left[\sum_{i=1}^{2}(x_i - z_i + z_i - y_i)^2\right]^{1/2}$

$$\leq \left(\sum_{i=1}^{2}(x_i - z_i)^2\right)^{1/2} + \left(\sum_{i=1}^{2}(z_i - y_i)^2\right)^{1/2}$$

(using Minkowski's Ineq.)

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$= d(x, z) + d(z, y)$

thus $d(x, y) \leq d(x, z) + d(z, y)$

**EXERCISE:** Show that (X, d) is a metric space where, $X = R^3$ , $d(x, y) = \left[\sum_{i=1}^{3}(x_i - y_i)^2\right]^{1/2}$

**EXAMPLE:** (4) If d is metric on a set $X \neq \phi$ then $d^*(x, y) = \min\{1, d(x, y)\}$ is also a metric on X.

SOLUTION:

Given that d is a metric on X so we have -

(i) $d(x, y) \geq 0$

(ii) $d(x, y) = 0 \Leftrightarrow x = y$

(iii) $d(x, y) = d(y, x)$

(iv) $d(x, y) \leq d(x, z) + d(z, y)$ , $\forall$ x, y, z ϵ X

now to show $d^*$ is metric on X

(i)* since $1 > 0$ and $d(x, y) \geq 0$ $\forall$ x, y ϵ X { using (i)}

$\Longrightarrow \min\{1, d(x, y)\} \geq 0 \Longrightarrow d^*(x, y) \geq 0$

(ii)* $d^*(x, y) = 0 \Leftrightarrow \min\{1, d(x, y)\} = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$ {since $1 \neq 0$ and using (ii)}

(iii)* $d^*(x, y) = \min\{1, d(x, y)\} = \min\{1, d(y, x)\} = d^*(y, x)$ {using (iii)}

(iv)* Let z ϵ X from $d^*(x, y) = \min\{1, d(x, y)\}$ we have $d^*(x, y) \leq d(x, y)$ $\forall$ x, y ϵ X ..... (#)

case I : if either $d(x, z) \geq 1$ or $d(z, y) \geq 1 \Longrightarrow$ either $\min\{1, d(x, z)\} = 1$ or $\min\{1, d(z, y)\} = 1$

$\Longrightarrow$ either $d^*(x, z) = 1$ or $d^*(z, y) = 1$

so $d^*(x, z) + d^*(z, y) \geq 1 \geq d^*(x, y)$

case II: if $d(x, z) < 1$ and $d(z, y) < 1$ then $\min\{1, d(x, z)\} = d(x, z)$ and $\min\{1, d(z, y)\} = d(z, y)$

$\Longrightarrow d^*(x, z) = d(x, z)$ and $d^*(z, y) = d(z, y)$ ...... (##)

now, $d^*(x, y) \leq d(x, y)$ (using equation (#))

$\leq d(x, z) + d(z, y)$ (using (iv))

$= d^*(x, z) + d^*(z, y)$ (using equation (##))

thus, in both cases $d^*(x, y) \leq d^*(x, z) + d^*(z, y)$, $\forall$ x, y, z ϵ X

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**OPEN SPHERE ( OPEN BALL):** Let X be a non empty set and d is a metric on X then open sphere

of radius r > 0 ( positive real number) centred at $x_0 \epsilon$ X denoted and defined by-

$S(x_0, r) = S_r(x_0) = \{x \epsilon X: d(x_0, x) < r\}$

**CLOSED SPHERE ( CLOSED BALL):** Let X be a non empty set and d is a metric on X then open sphere

of radius r > 0 ( positive real number) centred at $x_0 \epsilon$ X denoted and defined by-

$S[x_0, r] = Sr[x_0] = \{x \epsilon X: d(x_0, x) \le r\}$

EXAMPLE (1): Find open and closed sphere in usual meric space (R, d).

SOLUTION:

Since we know that usual metric on R is defined by $d(x, y) = |x – y|$

let $x_0 \epsilon$ R be the centre and r > 0 be a radius.

Open shpere $S(x_0, r)= \{x \epsilon R: d(x_0, x) < r\} = \{x \epsilon R : |x_0 - x|< r\}$ ( by def. of d)

$= \{x \epsilon R : | x - x_0|< r\} = \{x \epsilon R : - r < x - x_0 < r\}$

$= \{x \epsilon R: x_0 - r < x < x_0 + r\} = (x_0 - r, x_0 + r)$


thus we can see that open sphere in usual metric on R is an open interval with end point $x_0 - r$ and $x_0 + r$.

Similarly, closed sphere $S[x_0, r] = \{x \epsilon R: x_0 - r \le x \le x_0 + r\} = [x_0 - r, x_0 + r]$ is closed


EXAMPLE (2) Find open and closed sphere in discrete metric space (X, d).

SOLUTION:

<u>To find open sphere</u> : given that d is a discrete metric on X i.e.

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \quad \forall\ x, y \ \epsilon X \end{cases}$$

let $x_0 \epsilon$ X and r >0 then $S(x_0, r) = \{x \epsilon X: d(x_0, x) < r\}$

case I: if 0 < r < 1 then $S(x_0, r) = \{x \epsilon X: d(x_0, x) = 0\}$ because d(x, y) either
$\Longrightarrow S(x_0, r) = \{x \epsilon X: x = x_0 \} \Longrightarrow S(x_0, r) = \{ x_0 \}$ [using (ii) property of metric]

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case II: if $r \geq 1$ then $S(x_0, r) = \{x \in X: d(x_0, x) < 1\} = \{x \in X: d(x_0, x) = 0\} = \{x \in X: x = x_0\} = \{x_0\}$

thus $\forall$ $r > 0$, every singleton is an open sphere in discrete metric space.

<u>To find closed sphere</u> :

Case I : if $0 < r < 1$ then $S[x_0, r] = \{x \in X: d(x_0, x) \leq r\} = \{x \in X: d(x_0, x) = 0\}$ [since $r < 1$ ]

$= \{x \in X: x = x_0\} = \{x_0\}$ [(ii) property]

Case II: if $r \geq 1$ then $S[x_0, r] = \{x \in X: d(x_0, x) \leq 1\} = X$ [ by def. of d ]

thus $\forall$ $r > 0$, every singleton and whole X are closed sphere in discrete metric space.

EXAMPLE (3) Find open and closed sphere of unit radius centred at origin in the metric space $(R^2, d)$ where $d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)^2]^{1/2}$ , $x = (x_1, x_2)$, $y = (y_1, y_2) \in R^2$

SOLUTION :

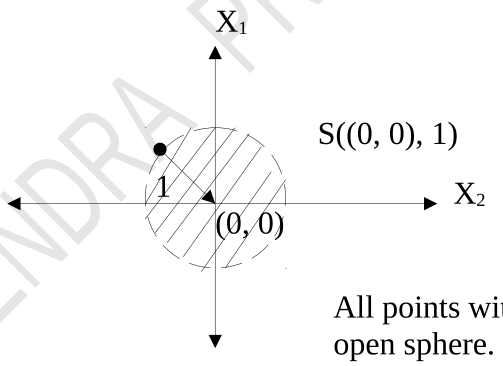
<u>To find open sphere</u>: given that centre $x_0 = (0, 0)$ and radius $r = 1$

so, $S(x_0, r) = \{x \in X: d(x_0, x) < r\} \Longrightarrow S((0, 0), 1) = \{ x \in X: d((0, 0), (x_1, x_2)) < 1 \}$

$= \{ x \in X: [(x_1 - 0)^2 + (x_2 - 0)^2]^{1/2} < 1 \}$

$= \{ x \in X: x_1^2 + x_2^2 < 1 \}$

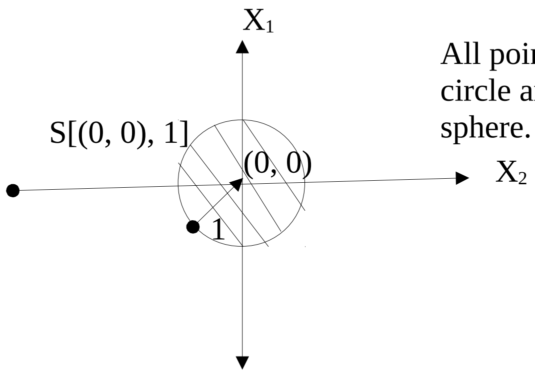
=set of all points within a circle of radius 1 centred at origin except poitns on circumference




All points within the circle are included in the open sphere.

<u>To find closed sphere</u>:

in the similar way, closed sphere , $S[(0, 0), 1] = \{ x \in X: x_1^2 + x_2^2 \leq 1 \}$




All points within and on the circle are included in closed sphere.

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**Exercixe:** Show that $(R^2, d)$, where $d(x, y) = |x_1-y_1| + |x_2-y_2|$ $\forall$ $x = (x_1, x_2), y = (y_1, y_2) \in R^2$.

Also find open and closed sphere of unit radius centred at origin in this metric space.

**NEIGHBOURHOOD (nbd) OF A POINT:** let (X, d) be a metric space and $N \subseteq X$ then N is said to be nbd of $x \in X$ if there exist an open sphere of centred at x of radius r > 0 such that

$S(x, r) \subseteq N$

**Remark:** If for all $r > 0$, $S(x, r) \not\subset N$ then N is not nbd of x.

**Exanple**:**1(a)**. In usual metric space on R, let $N = (1, 2) \subseteq R$ and x = 1.5 check whether N is a neighborhood of the point x or not.

solution: if we choose radius r = 0.1 >0 then open sphere centred at x = 1.5

$S(1.5, 0.1)$ $S(1.5, 0.1) = (1.5-0.1, 1.5+0.1) = (1.4, 1.6) \subseteq (1, 2) = N$

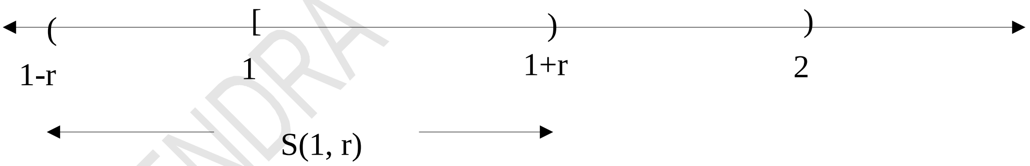
so (1, 2) is a nbd of 1.5

**1(b)**. If N = [1, 2) and x = 1 check N is a nbd of x or not.

solution: if we choose any radius r >0 then open sphere $S(1, r) = (1-r, 1+r)$ is not contained in [1, 2)

i.e. $(1-r, 1+r) \not\subset [1, 2)$

so [1, 2) is not a nbd of 1




**OPEN SET:** let (X,d) be a metric space and $G \subseteq X$ then is G is said to be an open set if it is nbd of its all points.

i.e. G is said to be an open set of X if $\forall$ $x \in G$ there exits r >0 such that $S(x, r) \subseteq N$.

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**EXAMPLES:** In usual metric space on R, the following subsets of R are open or not ?

(a) N ( set of natural numbers)

(b) Z ( set of integers)

(c) Q ( set of rational numbers)

(d) $Q^c$ (set of irrational numbers)

(e) R ( set of real numbers)

**Solution:** (a) Let x be any natural no. then for any r >0 $S(x, r) = (x-r, x+r) \not\subset N$.

because (x-r, x+r) also contains non-natural nos.Thus N is not nbd of its all points , hence **not an open set.**

(b) Let x be any integer no. then for any r >0 $S(x, r) = (x-r, x+r) \not\subset Z$

because (x-r, x+r) also contains non-integer nos.Thus Z is not nbd of its all points , hence **not an open set.**

(c) Let x be any rational no. then for any r >0 $S(x, r) = (x-r, x+r) \not\subset Q$

because (x-r, x+r) also contains irrational nos.Thus Q is not nbd of its all points , hence **not an open set.**

(d) same as part (c)

(e) Let x be any real no. then there exists r >0 ( even for every r >0), $S(x, r) = (x-r, x+r) \subseteq R$ beacuse (x-r, x+r) contions infinitely many rational and irrational points, so contained in R. Thus R is nbd of its all its points, hence R is **an open set**.

**PROPERTIES OF OPEN SET:** Let (X, d) be a metric space-

**(a)** $\phi$ is always an open set in X

proof: $\phi$ is trivially an open set or we can say that there is no point in $\phi$ for which it is not a nbd, hence an open set.

**(b)** X is always open in X.

Proof: Let x be any point of X then there exists r >0 ( even for every r >0) s. t.

$S(x, r) \subseteq R$, hence X is always an open set.

**(c)**Union of arbitrary collection of open sets is open.

Proof: Let $\{ G_\lambda : \lambda \in I \}$, where I is an index set, be an arbitrary collection of open sets.

T<u>o show $\bigcup_{\lambda \in I} G_\lambda$ is an open set</u>, let $x \in \bigcup_{\lambda \in I} G_\lambda$ be an arbitrary point

$$\Longrightarrow x \in G_{\lambda 0}, \quad \text{for some } \lambda_0 \in I$$

since $G_{\lambda 0}$ is an open set so it is a neighbourhood of each of its points

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so there exist r >0 such that $x \in S(x, r) \subseteq G_{\lambda 0}$, for some $\lambda_0 \in I$

$$\Longrightarrow S(x, r) \subseteq \bigcup_{\lambda \in I} G_\lambda \qquad \text{Since } G_{\lambda 0} \subseteq \bigcup_{\lambda \in I} G_\lambda$$

$$\Longrightarrow \bigcup_{\lambda \in I} G_\lambda \text{ is a nbd of } x$$

since x is arbitrary so $\bigcup_{\lambda \in I} G_\lambda$ is a nbd of each of its points

hence $\bigcup_{\lambda \in I} G_\lambda$ is an open set.

**(d)** Intersection of finite collection of open sets is open.

In particular, let $G_1$ and $G_2$ be two open set then $G_1 \cap G_2$ is open.

Proof: to show $G_1 \cap G_2$ is an open set, let $x \in G_1 \cap G_2$ be an arbitrary point

$\Longrightarrow x \in G_1$ and $x \in G_2$

since $G_1$ and $G_2$ are open so there exist $r_1 > 0$ and $r_2 > 0$ s.t. $x \in S(x,r_1) \subseteq G_1$ and $S(x,r_2) \subseteq G_2$

choose $\min\{r_1, r_2\} = r$ (say)>0

then $S(x,r) \subseteq G_1$ and $S(x,r) \subseteq G_2 \Longrightarrow \quad \subseteq S(x,r) \quad G_1 \cap G_2$

since x is arbitrary , so we can say that for all $x \in G_1 \cap G_2$ there exists r>0 s.t. $S(x,r) \subseteq G_1 \cap G_2$

$\Longrightarrow G_1 \cap G_2$ is nbd of its all points

$\Longrightarrow G_1 \cap G_2$ is an open set.

**Remark:** Intersection of arbitrary collection of open sets need not be an open set.

Example: Consider usual metric on R and $\{ (-1/n, 1/n) : n \in N \}$ be collection of open sets in R as open intervals in usual metric are open .

$$\text{Now,} \qquad \bigcap_{n \in N} (-1/n, 1/n) = \{0\}$$

to show {0} is not open, let $0 \in \{0\}$ then for any r >0, (0-r, 0+r) = (-r, +r) $\not\subset$ {0}

$\Longrightarrow$ {0} is not nbd of 0, hence not an open set.

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**THEROEM:** In any metric space, every open sphere is an open set.

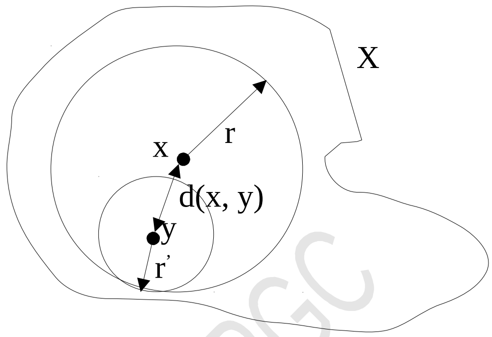
Proof:Let be (X, d) be a metric space and S(x, r) be an open sphere centred at x of radius r >0

$S(x, r) = \{ y \in X : d(x, y) < r \}$

Let $y \in S(x, r)$ be an arbitrary point

so $d(x, y) < r \implies r - d(x, y) > 0$

let $r' = r - d(x, y)$ ............ (i)




to show $S(y, r') \subseteq S(x, r)$

let $z \in S(y, r') \implies d(z, y) < r'$

$\implies d(z, y) < r - d(x, y)$ [ using (i) ]

$\implies d(z, y) + d(x, y) < r$ or $d(x, y) + d(y, z) < r$ [since $d(z, y) = d(y, z)$]

$\implies d(x, z) < r \implies z \in S(x, r) \implies S(y, r') \subseteq S(x, r)$

Thus S(x, r) is a nbd of y which is arbitrary, hence S(x, r) is nbd of its all points.

therefore S(x, r) is an open set.

**THEOREM:** Let (X, d) be a metric space and G be a subset of X then G is open if and only if G is union of open spheres.

Proof: Case I: Let G is open

$\implies$ G is nbd of each of its points i.e. $\forall$ $x \in G$ there exists $r_x > 0$ s. t. $S(x, r_x) \subseteq G$ so

$$\bigcup_{x \in G} S(x, r_x) \subseteq G \quad .......(i)$$

since $x \in S(x, r_x) \implies \subseteq \{x\} \quad S(x, r_x)$

$$\implies \bigcup_{x \in G} \{x\} \subseteq \bigcup_{x \in G} S(x, r_x) \implies G \subseteq \bigcup_{x \in G} S(x, r_x) \quad .....(ii)$$

$$\text{from (i) and (ii), } G = \bigcup_{x \in G} S(x, r_x)$$

$\implies$ G is Union of open spheres.

Case II: <u>let G is Union of open spheres.:</u>

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$$\Longrightarrow G = \bigcup_{x \in G} S(x, r_x)$$

Since we know that open is sphere is an open set and arbitrary Union of open sets is open,
hence G is open.

**CLOSED SET:**

Let (X,d) be a metric space then a subset F of X is said to be closed if its complement i.e. $F^c$ is open.

Examples:In usual metric space on R

(i) [a, b] is closed.

since $[a, b]^c = (-\infty, a) \cup (b, \infty)$

here, $(-\infty, a)$ and $(b, \infty)$ are open so $(-\infty, a) \cup (b, \infty)$ is open

hence $[a, b]^c$ is open $\Longrightarrow$ [a, b] is closed.

(ii) Set of natural no. N is closed.

since N = {1, 2, 3,........} and $N^c = R - N = (-\infty, 1) \cup (1, 2) \cup (2, 3) \cup$ ..........

$$= (-\infty, 1) \cup \bigcup_{n=1}^{\infty} (n, n+1)$$

since for all n ϵ N , (n, n+1) is open also $(-\infty, 1)$ is open and we know that arbitrary union of open sets is open , hence

$$(-\infty, 1) \cup \bigcup_{n=1}^{\infty} (n, n+1) \quad \text{is open}$$

So $N^c$ is open therefore **N is closed.**

(iii) Similarly we can show **Z ( set of all integers) is closed**.

(iv)**Q and $Q^c$ are not closed** as its complements are not open.

( refer to examples of open set)

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**PROPERTIES OF CLOSEDSET:** Let (X, d) be a metric space then-

**(a).** $\phi$ set is always closed in X

Proof: since we $\phi^c = X$ and X is open ( refer to property of open set)

so $\phi^c$ is open , hence $\phi$ is closed.

**(b).** X is always closed in X

since $X^c = \phi$ and $\phi$ is open, so $X^c$ is open and hence X is closed.

**(c).** Intersection of arbitrary collection of cloesd sets is closed.

Proof: Let { $F_\lambda : \lambda \in I$ }, where I is an index set, be a collection of closed sets.

<u>to show $\bigcap_{\lambda\in I} F_\lambda$ is a closed set</u>,

$$\left(\bigcap_{\lambda\in I} F_\lambda\right)^c = \bigcup_{\lambda\in I} F_\lambda^{\ c}$$ [ Using Demorgan's law ]

since $F_\lambda$ is closed for all $\lambda \in I$, so $F_\lambda^c$ is open for all $\lambda \in I$

also we know that union of arbitrary collection of open sets is open

$$\text{so} \bigcup_{\lambda\in I} F_\lambda^{\ c} \text{ is open} \implies \left(\bigcap_{\lambda\in I} F_\lambda\right)^c \text{ is open}$$

$$\implies \bigcap_{\lambda\in I} F_\lambda \text{ is closed}$$

**(d).**Union of finite collection of closed sets is closed.

In particular, let $F_1$ and $F_2$ are two closed sets then intersection of $F_1$ and $F_2$ is closed.

Proof: by Demorgan's law, $( F1 \cup F2 )^c = F_1^c \cap F_2^c$

since $F_1$ and $F_2$ are closed $\implies$ $F_1^c$ and $F_2^c$ are open

$\implies F_1^c \cap F_2^c$ is open [ since finite intersection of open sets is open ]

$\implies$ $\cup$(F1 F2 )$^c$ is open $\cup \implies$ F1 F2 is closed.

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**Remark:** Union of arbitrary collection of closed sets need not be closed.

Example: In usual metric space on R, consider the collection { [1/n , 1] : n ϵ N } so

$$\bigcup_{n\in N}[1/n,1] = (0\ ,\ 1]$$

now, since $(0, 1]^c = (-\infty, 0] \cup (1, \infty)$

here $(-\infty, 0]$ is not nbd of 0 [ check it ]

$\Longrightarrow$ $(-\infty, 0]$ is not nbd of its all points hence not open

$\Longrightarrow$ $(-\infty, 0] \cup (1, \infty)$ is not open $\Longrightarrow$ $(0, 1]^c$ is not open $\Longrightarrow$ $(0, 1]$ is not closed

hence the statement.

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