Electrostatics

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Q. What is electrostatics?

Ans. When all the charges present in a region are not moving with time then it is said to be electrostatics. It involves calculation of electric field, electric force, electric potential and so many other properties of a charge distribution. Electric charges are of two types, positive electric charge and negative electric charge.

Q. What is the origin of charges?

Ans. In an atom the negatively charged electrons revolve around a positively charged nucleus. If we remove one or more electrons from an atom it becomes positively charged ion, while addition of one or more electrons makes it negatively charged ions. Generally outermost electrons from an atom are removed. An atom is neutral because it contains equal number of electrons and protons. The charge on an electron or proton is fundamental; charges on other particles are multiple of these charges.

Fundamental Rule of interaction:

Like charges repel each other while opposite charges attract each other.

Conservation of charge

Charges are never created nor destroyed but they are simply transferred from one material to other material.

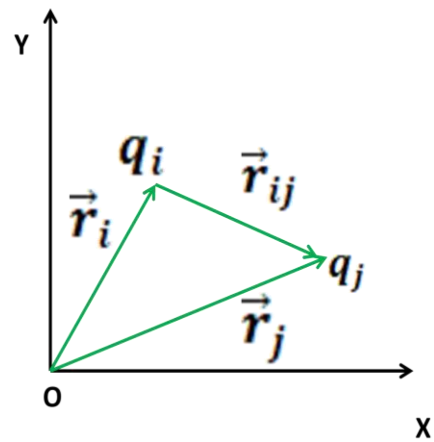
Based on electrical conductivity we can have following materials

(a) Conductors: Those materials which conduct electricity
(b) Insulators: Materials which do not conduct electricity
(c) Semiconductors: Those materials which conduct electricity at room temperature but not at absolute zero. Their conductivity can be engineered.

Coulomb's Law

The Coulomb's law states that the force between two charges is

(1) Proportional to product of the charges and
(2) Inversely proportional to the distance between the charges
(3) The direction of the force is along the line joining the two charges



So the force between point charges q_i and q_j can be given as

$$\overrightarrow{F_{ij}} = \frac{k q_i q_j \hat{r}_{ij}}{r^2{}_{ij}},$$

Where $\vec{r}_{ij}$ is the vector joining q_i and q_j and $k = \frac{1}{4\pi\epsilon}$ and ϵ is the permittivity of the medium

For air, $\epsilon=\epsilon_0=8.854*10^{-12}$ $Coul^2/Nm^2$

For a system of n charges the force on j^{th} charge is given by

$$\vec{F_j} = \sum \vec{F}_{ij} = \frac{1}{4\pi\epsilon} q_j \sum_{i \neq j} \frac{q_i}{r^2{}_{ij}} \hat{r}_{ij}$$

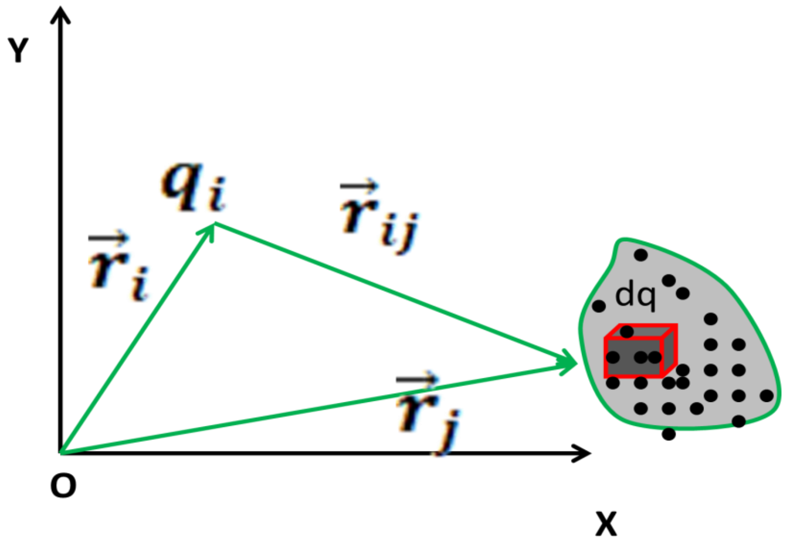
For continuous charge distribution in a volume "V" we need to define the volume charge density(ρ), surface charge density(σ) and length charge density λ_l as

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V}\ , \quad \sigma = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} \quad \text{and} \quad \lambda = dQ/dL$$

So, we have to calculate force $\vec{F}$ due to continuous charge distribution dq on a point charge q_i, here we have to integrate over charge distribution dq

$$\vec{F} = \frac{q_i}{4\pi\epsilon_0} \int \frac{dq}{r_{ij}{}^3} \vec{r}_{ij} = \frac{q_i}{4\pi\epsilon_0} \int \frac{\vec{r}_{ij}}{r_{ij}{}^3} \rho(r_j) dV + \frac{q_i}{4\pi\epsilon_0} \int \frac{\vec{r}_{ij}}{r_{ij}{}^3} \sigma(r_j) dS$$

Where, $\vec{r}_{ij}$ is a vector joining charge q_i and charge at j^{th} position. And r_j is the general position of charge in the continuous charge distribution



The Electric Field Strength:

The presence of one electric charge affects other charge around it, this effect is due to the electric field of original charge. The electric field strength can be defined mathematically as

$$\vec{E} = \frac{\vec{F}}{q_0}$$

Where q is an infinitesimal charge

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r}$$

Thus the electric field at $\vec{r}$ due to n discrete charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n} q_i \frac{(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3}$$

While for a continuous charge distribution

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3} \rho(\vec{r}_j) dV + \frac{1}{4\pi\epsilon_0} \int \frac{(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3} \sigma(\vec{r}_j) dS$$

Thus we can write electric field due to a continuous line, surface and volume charge density as $\vec{E} = \int \frac{\lambda_l}{4\pi\epsilon} \frac{\hat{r}}{R^2} dL$, $\vec{E} = \int \frac{\sigma dS}{4\pi\epsilon} \frac{\hat{r}}{R^2}$ and $\vec{E} = \int \frac{\rho}{4\pi\epsilon} \frac{dV}{R^2} \hat{r}$

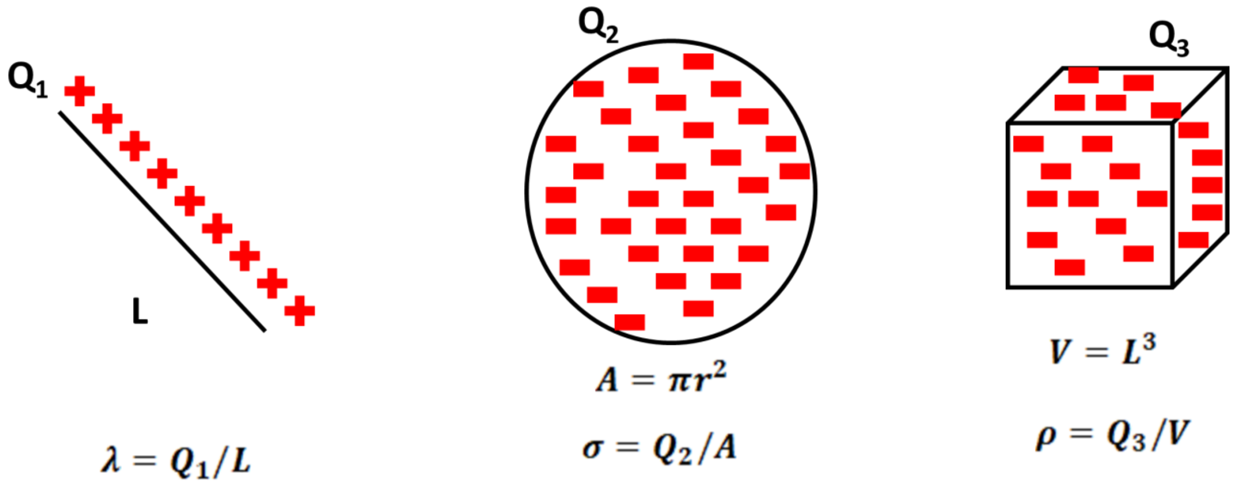
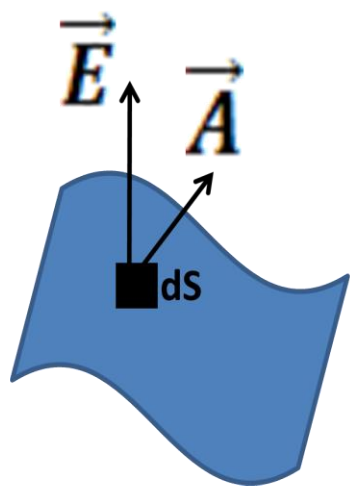


Figure 1: The line, surface and volume charge density

Electric flux: The total electric field lines coming out from any surface is called flux of electric field through that surface.

Flux, $\phi = \vec{E}.\vec{A}$

Where $\vec{A}$ is vector area of the surface and $\vec{E}$ is electric field strength



For a general surface the differential flux is given as follow

$$d\phi = \vec{E}.d\vec{S}$$

Or $\phi = \int \vec{E}\,.d\vec{S}$

The direction of the $d\vec{S}$ is along the normal to the surface

Gauss's Law: The electric flux flowing outside through any closed surface of any shape is $1/\epsilon_0$ times of the total charge enclosed inside the surface,

$$\oint \vec{E}\,.\hat{n}dS = \frac{Q_{enc}}{\epsilon_0}$$

Proof: For a point charge q, the electric field at a point P(r) is given by

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

So
$$\vec{E}(r).\hat{n}dS = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}.\hat{n}dS$$

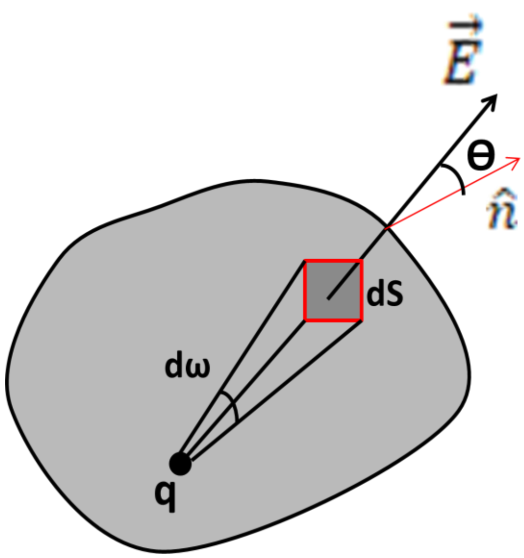
Or
$$\oint \vec{E}(r).\hat{n}dS = \frac{q}{4\pi\epsilon_0}\oint\frac{\hat{r}.\hat{n}}{r^2}dS$$

But $\frac{\hat{r}.\hat{n}}{r^2}dS$ is actually solid angle subtended by the infinitesimal surface element dS on charge q.

$$\therefore \oint\frac{\hat{r}.\hat{n}}{r^2}dS = total\ solid\ angle\ subtended\ by\ all\ dS\ at\ charge\ q = 4\pi$$

$$\oint \vec{E}(r).\hat{n}dS = \frac{q}{4\pi\epsilon_0}4\pi = \frac{q}{\epsilon_0}$$

$$\oint \vec{E}(r).\hat{n}dS = \frac{q}{\epsilon_0}$$



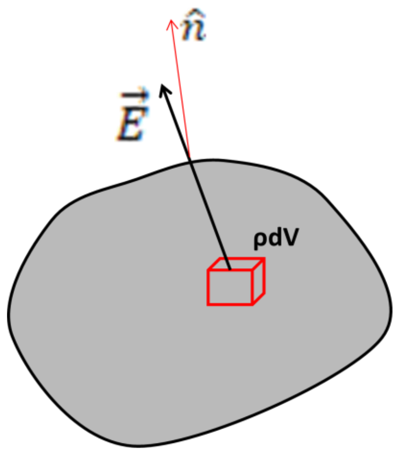


Figure 2: For continuous charge distribution ρdV may be considered a point charge

If the surface encloses a volume charge density ρ inside it, then a very small charge (point charge) ρdV gives $\frac{1}{\epsilon_0}\rho dV$ to normal component of flux. For a bigger surface enclosing the whole continuous charge distribution

$$\oint \vec{E}\,.\hat{n}dS = \frac{1}{\epsilon_0}\int \rho\, dV$$

From Gauss's Divergence theorem, we can write, $\oint \vec{E}\,.\hat{n}dS = \int(\nabla.\vec{E})dV$

Thus $$\int(\nabla.\vec{E})dV = \frac{1}{\epsilon_0}\int \rho\, dV,$$

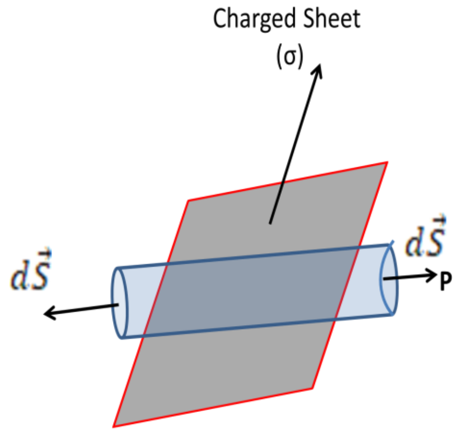
$$\therefore\ \nabla.\vec{E} = \frac{\rho}{\epsilon_0}$$

This is the differential form of Gauss's Law also known as first Maxwell's equation

Applications of Gauss's Law

The applications of Gauss's law are in finding the r dependency of electric field while knowing the symmetry of the surface.

(1)***Electric field at a point due to A Thin charge Sheet***



Here we want to calculate the electric field at point P due to the charged sheet (σ). So first of all we draw a Gaussian Surface across the sheet and calculate the flux of electric field passing through it.

$$\phi = \phi_{curved\ surface} + \phi_{plane\ surface}$$

But $\phi_{curved\ surface} = 0$ (because $\vec{E}$ is perpendicular to the curved surface always)

$$\therefore \phi = \phi_{plane\ surface}$$

$$\phi = \vec{E}.\vec{S} + \vec{E}.\vec{S}$$

$$\phi = ES\ Cos(0) + ES\ Cos(0) = 2ES$$

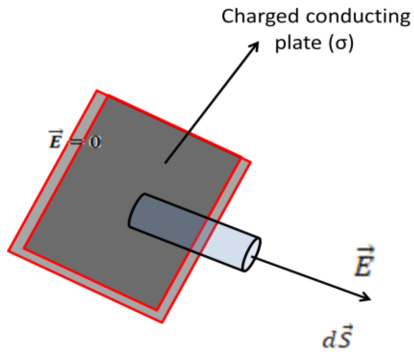
$$\phi = 2ES = 2E\pi r^2$$

Now charge enclosed by this Gaussian Surface is $Q_{enc} = \sigma S = \sigma(\pi r^2)$

$$\therefore 2E\pi r^2 = \frac{\sigma\pi r^2}{\epsilon_0}$$

Or $E = \sigma/2\epsilon_0$

(2) ***Electric field due to a charged conducting plate***



Again $\phi_{curved\ surface} = 0$, because the $\vec{E}$ is perpendicular to the curved surface of Gaussian surface.

$$\phi_{plane\ surface} = \vec{E}.\vec{S}$$

$$= ESCos(0) = E\pi r^2$$

Charge enclosed $Q_{enc} = \sigma S = \sigma\pi r^2$

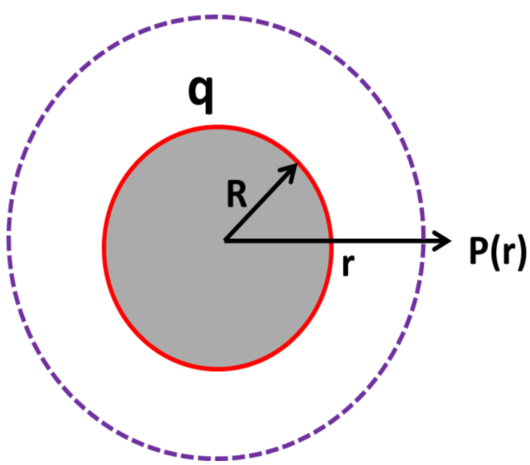
So from Gauss Law, $E\pi r^2 = \frac{\sigma\pi r^2}{\epsilon_0}$

$$E = \frac{\sigma}{\epsilon_0}$$

(3) ***Uniformly charged sphere***

Here we consider a sphere with total charge q, radius R and charge density ρ.

For points outside the sphere



If the charge is uniformly distributed with volume charge density ρ, then

$\oint \vec{E}\,.\hat{n}dS = E(4\pi r^2)$, because the electric field is radial and area vector of Gaussian surface is also radial.

Since the point is outside the charged sphere, So $Q_{enc} = q$

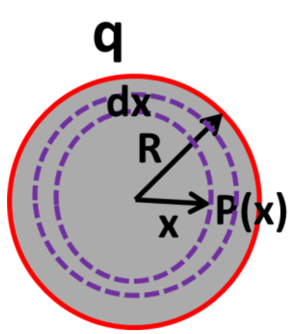
$$E(4\pi r^2) = \frac{q}{\epsilon_0}$$

$$\vec{E}(r) = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

For points on the surface, r=R

$$\vec{E}(r = R) = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{R^2}$$

For points inside the sphere,



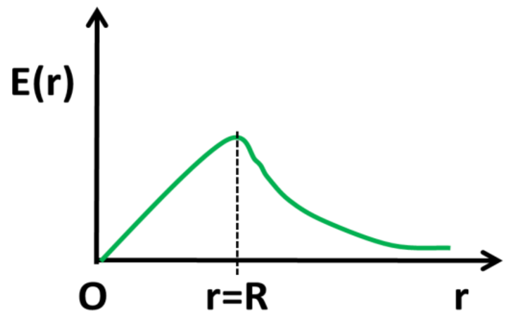
$$\frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_0^r \rho\, 4\pi x^2 dx$$

Where, $\rho = \frac{q}{\frac{4}{3}\pi R^3}$

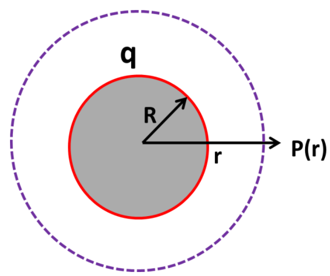
$$\frac{Q_{enc}}{\epsilon_0} = \frac{q}{\epsilon_0}\frac{r^3}{R^3}$$

$$\therefore \; E(4\pi r^2) = \frac{q}{\epsilon_0}\frac{r^3}{R^3}$$

Or $$\vec{E}(r) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{R^3}$$



Isolated Uniformly Charged Spherical conductor (For charge on outer surface only)



For points inside the spherical conductor

$$\oint \vec{E}\,.\hat{n}dS = \frac{Q_{enc}}{\epsilon_0},$$

And since the charge is uniformly distributed on the surface of conductor, So $Q_{enc} = 0$

$$\vec{E} = 0$$

For points outside the spherical conductor $Q_{enc} = q$

For spherical Gaussian Surface around the spherical conductor

$$E 4\pi r^2 = \frac{q}{\epsilon_0}$$

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

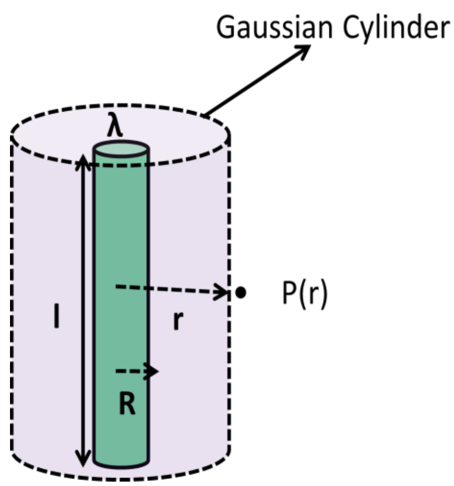
At surface, r=R

$$\vec{E}(R) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{r}$$

Charged cylindrical conductor

For points outside the cylinder

(the line charge density is λ and length of the conductor is l)



For the given Gaussian surface the Gauss's law can be written as

$$\oint \vec{E}.\hat{n}dS = \frac{Q_{enc}}{\epsilon_0}$$

Since the $\vec{E}$ is radial and $\hat{n}$ is also radial

$$\therefore E(2\pi r l) = \frac{\lambda l}{\epsilon_0}$$

$$Or\ \ \vec{E}(r) = \frac{\lambda}{2\pi\epsilon_0}\frac{\hat{r}}{r}$$

For uniform distribution of charge throughout the volume(r<R) and the volume charge density is ρ, then

$$\oint \vec{E}.\hat{n}dS = \frac{Q_{enc}}{\epsilon_0}$$

$$E(2\pi r l) = \frac{\rho(\pi R^2 l)}{\epsilon_0}$$

$$\vec{E}(r) = \frac{\rho R^2}{2\epsilon_0}\frac{\hat{r}}{r}$$

<u>For charge only at the Surface</u>

The electric field inside the cylinder will be zero and for outside the cylinder

$$\oint \vec{E}\,.\,d\vec{S} = \frac{Q_{enc}}{\epsilon_0}$$

Due to symmetry of the Gaussian surface and radially outward direction of electric field and $d\vec{S}$ we can write

$$E(2\pi r l) = \frac{\pi r^2 l \rho}{\epsilon_0}$$

Or

$$\vec{E}(r) = \frac{\vec{r}\rho}{2\epsilon_0}$$