ADVANCED LINEAR ALGEBRA

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EIGEN VECTOR AND EIGEN VALUE

1.1 <u>Definition</u>: Let V be a vector space over the field F ($F = \mathbb{R}$ or $\mathbb{C}$). Let $T : V \longrightarrow V$ be a linear operator. A nonzero vector $v \in V$ is called an eigenvector of T if there exists a scalar λ such that $T(v) = \lambda v$.

The scalar λ is called the eigenvalue corresponding to the eigenvector v.

1.2. <u>Definition</u>: Let A be an $n \times n$ matrix with entries in the field F. A nonzero vector $v \in F^n$ is called an eigenvector of A if $Av = \lambda v$ for some scalar λ. The scalar λ is called the eigenvalue of A corresponding to the eigenvector v.

1.1' <u>Definition</u>: Let $T : V \longrightarrow V$ be a linear operator on a vector space V over the field F. A scalar λ is called an eigenvalue of T if there exists a non zero vector $v \in V$ such that

$$T(v) = \lambda v. \qquad \text{———} (1)$$

Any vector satisfying (1) is called an eigenvector of T corresponding to the eigenvalue λ.

Note that any scalar multiple of an eigenvector is again an eigenvector: for if $T(v) = \lambda v$ & $k \in F$, then

$$T(kv) = k(T(v))$$

$$= k(\lambda v) = \lambda(kv).$$

$kv \in V$

Example Set (1 – 5)

Example 1. Consider the identity map I on V. Then by definition

$$I : V \longrightarrow V$$
$$v \rightsquigarrow I(v) = v \quad \forall\, v \in V.$$

This gives

$$I(v) = v = 1.v \quad \forall\, v \in V;$$

i.e. every vector in V is an eigenvector corresponding to the eigenvalue 1.

Example 2. Consider the function $T : R^2 \longrightarrow R^2$ given as

$$T(x,y) = (x+y,\, x+y). \quad \forall\, x, y \in R.$$

Then
$$T(1,1) = T(2,2)$$
$$= 2(1,1)$$
$$\Rightarrow 2 \text{ is an eigen value of } f$$

and $(1,1)$ is an eigen vector of T.

Example 2. Consider the function $T : R^2 \longrightarrow R^2$ given as

$$T(x,y) = (x+y,\ x+y). \quad \forall\ x, y \in R.$$

Then
$$\begin{aligned} T(1,1) &= \cancel{T}(2,2) \\ &= 2(1,1) \end{aligned}$$

$\Rightarrow$ 2 is an eigen value of f and $(1,1)$ is an eigen vector of T.

Example 3. Let the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \quad \text{and} \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in R^2$$

Then
$$A v_1 = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 4+(-2) \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$= -2 \begin{pmatrix} +1 \\ -1 \end{pmatrix} = -2 v_1.$$

Shows that -2 is an eigenvalue and $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of A.

Example 4. $T : R^2 \longrightarrow R^2$ be a linear operator that rotates each vector in the plane through the ~~vector~~ angle of $\pi/2$. Then there exist no non zero vector $v \in R^2$

s.t. $T(v) = \lambda v$, for any λ.

So T has no eigenvalue and no eigenvector.

NOTE: There exist operators (and matrices) with no eigenvalues or eigenvectors.

Example 5. Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Find the eigenvalues and associated non-zero eigenvectors.

We shall find a scalar λ and a non-zero vector $v = \begin{pmatrix} x \\ y \end{pmatrix}$ such that

$$Av = \lambda v.$$

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Now,

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

or
$$x + 2y = \lambda x$$
$$3x + 2y = \lambda y$$

or
$$\left.\begin{aligned} (1-\lambda)x + 2y &= 0 \\ (3\cancel{-\lambda}x) + (2-\lambda)y &= 0. \end{aligned}\right\}$$

This system has a solution iff

or
$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

or $\quad 2-2\lambda-\lambda+\lambda^2-6=0.$

or $\quad \lambda^2-3\lambda-4=0$

or $\quad (\lambda-4)(\lambda+1)=0$

Thus $\lambda=4$ and $\lambda=-1$ are the eigenvalues of A.

<u>Eigenvectors Corresponding to $\lambda=4$</u> –

From (1), putting $\lambda=4$, we get

$$-3x+2y=0$$

and $\quad 3x-2y=0$

or $\quad 3x=2y$

This gives
$$v=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\3\end{pmatrix}\neq 0.$$

Thus $v=\begin{pmatrix}2\\3\end{pmatrix}\in R^2$ is a non zero vector such that

$$Av=4v.$$

i.e. $v=\begin{pmatrix}2\\3\end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda=4$ for the matrix A.

Eigenvectors corresponding to $\lambda = -1$ -

From (1) putting $\lambda = -1$, we obtain

$$2x + 2y = 0$$

and $$3x + 3y = 0$$

or $$x + y = 0$$

or $\quad y = -x.$

Thus

$$u = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in R^2 \text{ is a non zero vector}$$

such that

$$Au = -1 \cdot u.$$

i.e. $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda = -1$ of the matrix A.

—————— x ——————.

1.2 <u>Theorem</u>. Let $T: V \longrightarrow V$ be an linear operator on a vector space V over the field F. Then $\lambda \in F$ is an eigenvalue of T if and only if the operator $T - \lambda I$ is singular (not invertible).

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<u>Proof</u> Let $T : V \longrightarrow V$ be a linear operator. By definition, we know that a scalar $\lambda \in F$ is an eigenvalue of T if and only if there exists a non zero vector $v \in V$ such that

$$T(v) = \lambda v.$$

or $$T(v) = \lambda \cdot I(v), \quad I \text{ is an identity operator on } V.$$

or $$(T - \lambda \cdot I)(v) = 0 \qquad v \neq 0$$

or $T - \lambda \cdot I : V \longrightarrow V$ is a singular operator $\left(\operatorname{Ker}(T - \lambda I) \neq \{0\} \text{ as } v \in \operatorname{Ker}(T - \lambda \cdot I),\ v \neq 0\right)$

1.3 Remark: (Matrix Version)

Let A be an $n \times n$ matrix over the fiel
i.e. with entries in F. Then a scalar λ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

Above remark is based on the following theorem:

1.4. Theorem: Let A be an $n \times n$ matrix. Then the follow
are equivalent –

i) A is invertible i.e. A has an inverse A^{-1};

ii) A is non singular i.e. $AX = 0$ has only the zer
solution or rank $A = n$ or the rows (columns
A are linearly independent;

iii) $\det A = |A| \neq 0$.

1.5 **Definition [characteristic Polynomial]**

Consider an $n \times n$ matrix over the field F. Then the matrix $\lambda I_n - A$ where I_n is the $n \times n$ identity matrix and λ is an indeterminant, is called the characteristic matrix of A.

Its determinant $\Delta_A(\lambda) = \det(\lambda I_n - A)$ is a polynomial in λ, and is called the characteristic polynomial of A.

Note that

$$\lambda I_n - A = \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & & & \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{pmatrix}$$

where $A = (a_{ij})_{n \times n}$.

We call the equation

$$\Delta(\lambda) = \Delta_A(\lambda) = \det(\lambda I_n - A) = 0$$

The characteristic equation for A.

1.6 Theorem: Let A be an $n \times n$ matrix over the field F. A scalar $\lambda \in F$ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial $\Delta(\lambda)$ of A.

<u>Proof</u> From Theorem 1.2, we know that $\lambda \in F$ is an eigenvalue of A iff $\lambda I - A$ is singular

iff $\det(\lambda I - A) = 0$

i.e. λ is a root of $\Delta(\lambda)$.